

# Gödel Type Metrics in Randall Sundrum Model

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## Abstract

Anisotropic cosmological models such as the Gödel universe and its extensions - Gödel type solutions, are embedded on a visible 3-brane in the Randall-Sundrum 1 model. The size of the extra dimension is stabilized by tuning the rotation parameter to a very small value so that hierarchy problem can be solved. A limiting case also yields the Randall-Sundrum 2 model. The rotation parameter on the visible brane turns out to be of order  $10^{-32}$ , which implies that visible brane essentially lacks rotation.

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# 1 Introduction

In last few years, interest in Gödel universe has increased, following the discovery, that it was an exact solution of the string theory [1, 2, 3]<sup>1</sup>. Since its discovery in 1949, Gödel universe attracted attention, due to unusual properties, such as, existence of Closed Time-like Curves (CTCs), existence of rotation and consequent anisotropy, and anti-Machian nature. In 1982 Birch [7] argued a case for observational evidence for universe rotation and anisotropy. The case for an anisotropic universe, was further developed by Jain and Ralston [8, 9]. Gödel, used a universe density figure of  $10^{-30}$  gm/cm<sup>3</sup> and arrived at an angular velocity of  $10^{-18}$  rad/sec. He considered galaxies as point particle's in a pressureless fluid permeating the universe. The local vorticity in his model equals universe's angular velocity. The available data at that time, gave galaxies an angular velocity  $\sim 10^{-14}$ . He commented that his model did not explain random distribution of angular velocity vectors, and the observed expansion of the universe, which would modify the predictions. From an observed anisotropy of polarization of a set of radio source data, Birch [7] gave a figure of  $10^{-13}$  rad/sec, for angular velocity of universe. Jain and Ralston have suggested Virgo cluster as the direction of this anisotropy [8, 9].

In an altogether different development in theoretical physics, Arkani-Hamed, Dimoupolous and Dvali [10] proposed a low scale gravity model in order to solve hierarchy problem by introducing the concept of large extra dimensions (LED). Randall and Sundrum [11] suggested introduction of a single extra dimension in a warped geometry. This had more advantages than ADD model, from phenomenologically and theoretical view points. It would be therefore of interest to examine, how an anisotropic solution, such as the Gödel universe, when used in Randall- Sundrum (RS) models [11, 12] will behave - primarily as a check for any signs of anisotropy on the visible brane. In this paper we come to the conclusion that a visible Gödel type brane embedded in RS model has vanishingly small rotation.

Gödel universe has the line element,

$$ds^2 = a^2(dx_0^2 - dx_1^2 - dx_3^2 + \frac{e^{2x_1}}{2}dx_2^2 + 2e^{x_1}dx_0dx_2) \quad (1)$$

which satisfies the Einstein equations for a uniform matter density,  $\rho =$

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<sup>1</sup>It is also worth mentioning that, in a different context [4, 5], a whole family of Goedel-like solutions was investigated by using the AdS/CFT correspondence.

$(8\pi Ga^2)^{-1}$ , zero pressure, and a cosmological constant,  $\Lambda = -4\pi G\rho$  (here  $G$  in Newton's gravitational constant). The corresponding angular velocity is:  $\omega = 2(\pi G\rho)^{1/2} = (\sqrt{2}a)^{-1}$ . Any circle in  $(x_1x_2)$ -plane with radius  $R_c = c/\omega$  (where  $c$  is speed of light) is a closed null curve (CNC). It acts as a horizon for particles starting from its center and moving along geodesics. Its interesting to compare Gödel horizon with Friedmann-Robertson-Walker (FRW) horizon. Whereas FRW is expanding at speed of light, Gödel horizon is rotating at speed of light. Circles of radii larger than that of CNC, are Closed Time-like Curves (CTCs). For very small universe densities, radii of the CTCs will be very large, and presence of CTCs will largely be immaterial to local physics.

If in Einstein's field equations for Gödel universe, one puts cosmological constant equal to zero, then one obtains a solution with non-zero pressure. The two situations may be expressed as,

$$\begin{aligned} 8\pi G p_{matter} = 8\pi G \rho_{matter} = \frac{a^2}{2}, \quad \lambda_{cosmological} = 0, \text{ or} \\ 8\pi G \rho_{matter} = a^2, p_{matter} = 0, \quad 8\pi G \lambda_{cosmological} = -\frac{a^2}{2}. \end{aligned} \quad (2)$$

The line element of Randall Sundram (RS) models,

$$ds^2 = e^{-2\sigma(y)} \eta_{\mu\nu} dx^\mu dx^\nu + dy^2 \quad (3)$$

has a structural compatibility with Gödel line element, in the sense that the pre-factor  $e^{-2\sigma(y)}$ , can be directly equated with pre-factor  $a^2$  in Gödel line element. This allows exponential variation of rotation, pressure and density in bulk direction, which makes the fluid anisotropic in bulk direction. This however, is not unusual in physics. For example, earth's atmosphere has gradients of pressure, density etc., in vertical direction, which counter balance earth's gravitational pull. In RS models particles gravitate from hidden brane towards visible brane. In the Gödel type branes embedded in RS models, pressure gradient in the bulk direction, of the 5-dimensional anisotropic fluid, provides an additional force, which pushes particles towards visible brane.

The solution to Einstein equations in presence of a 5-D bulk cosmological constant  $\Lambda$  in RS set-up is given by

$$\sigma(y) = \sqrt{-\frac{\Lambda}{24M^3}} |y| = k|y| \quad (4)$$

This solution is valid only if brane tensions and cosmological constants are related as follows

$$\Lambda = -24M^3k^2, \quad v_1 = -v_2 = 24M^3k \quad (5)$$

If  $k > 0$ , the brane 1 at  $y = 0$  has positive tension  $v_1$  and brane 2 at  $y = y_c$  has negative tension  $v_2$ . Starting from the 5-D action, after integrating over extra coordinate  $y$  yields the reduced effective 4-D plank scale given by this relation

$$M_{Pl}^2 = \frac{M^3}{k}(1 - e^{-2ky_c}) \quad (6)$$

Setting the visible brane at  $y = y_c$  where our standard model fields live, it is found that any mass parameter  $m_0$  on the visible 3-brane in the fundamental higher-dimensional theory will correspond to a physical mass

$$m = e^{-ky_c}m_0$$

TeV mass scales can be generated on the 3-brane at  $y = y_c$  due to the exponential factor present in the metric which fixes  $ky_c \approx 35$ . Next section we construct solutions to Einstein equations in RS set-up in presence of bulk fluid.

## 2 Solution In Presence of Anisotropic Bulk Fluid

Consider a metric given by,

$$ds^2 = e^{-2\sigma(y)}\hat{g}_{\alpha\beta}(x)dx^\alpha dx^\beta + dy^2, \quad (7)$$

where  $\hat{g}_{\alpha\beta}$  is the induced 4-D metric. By assuming that the bulk matter has no flow along y-direction, i.e.,  $u^5 = 0$ , where,  $u^5$  is the velocity vector along y-direction, we arrive at the following form for 5-D anisotropic energy momentum tensor of bulk fluid,

$$T^{AB} = \begin{pmatrix} T^{\alpha\beta} & 0 \\ 0 & P \end{pmatrix}, \quad (8)$$

where  $T^{\alpha\beta}$  is energy-momentum tensor of the 4-D perfect fluid, given by,

$$T^{ab} = Au^a u^b + Bg^{ab} \quad (9)$$

and

$$T^{33} = Cg^{33} , \quad (10)$$

where  $a, b$  run over 0, 1, 2. From the energy momentum conservation  $T^{5B};_B = 0$  we obtain the condition

$$P' = -2\sigma'(y)(-A + 3B + C - 4P) , \quad (11)$$

where prime stands for partial derivative with respect to  $y$ . The above equation is referred to as the 'hydrostatic equilibrium equation' of the bulk fluid, along the  $y$ -direction [13].

In RS model, 5-D Einstein equations are given by

$$R_{AB} - \frac{1}{2}g_{AB}R = \frac{1}{4M^3}(T_{AB} - \lambda g_{AB} - v_1\delta(y) - v_2\delta(y - y_c)) . \quad (12)$$

Substituting the metric Eq. (7) into Eq. (12) and using the relation Eq. (5), we get the 4-D Einstein equations,

$$R_{\alpha\beta}^{(4)} - \frac{1}{2}g_{\alpha\beta}R^{(4)} = \frac{1}{4M^3}T_{\alpha\beta} \quad (13)$$

where the four dimensional Ricci scalar is given as

$$R^{(4)} = -\frac{1}{2M^3}Pe^{-2\sigma(y)} . \quad (14)$$

As can be seen right hand side of equations (13) and (14) depend upon the coordinate  $y$  which implies that these are not four dimensional tensors. However by relating 4-D energy momentum tensor to 5-D energy momentum tensor, the effective 4-D Einstein equation on the visible brane in presence of anisotropic perfect fluid source, can be written as,

$$R_{ab}^{(4)} - \frac{1}{2}g_{ab}R^{(4)} = 8\pi G_4 T_{ab}^{(4)} , \quad (15)$$

$$R_{33}^{(4)} - \frac{1}{2}g_{33}R^{(4)} = 8\pi G_4 T_{33}^{(4)} , \quad (16)$$

where  $T_{ab}^{(4)}$  and  $T_{33}^{(4)}$  are given by,

$$T_{ab}^{(4)} = \hat{A}\hat{u}_a\hat{u}_b + \hat{B}\hat{g}_{ab} \quad (17)$$

and

$$T_{33}^{(4)} = \hat{C} \hat{g}_{33} , \quad (18)$$

where

$$A = c_0 e^{2\sigma(y)} \hat{A}, \quad B = c_0 e^{2\sigma(y)} \hat{B}, \quad C = c_0 e^{2\sigma(y)} \hat{C} \quad (19)$$

$$u_a = e^{\sigma(y)} \hat{u}_a \quad (20)$$

and  $c_0 = 32M^3 \pi G_4$ .  $A, B$  and  $C$  satisfy the relation

$$2P = -A + 3B + C . \quad (21)$$

### 3 Gödel Type Solutions On The Visible brane

As indicated in the previous section, one can embed any solution of the four dimensional Einstein equations with perfect fluid source, in RS models. One of option is the Gödel solution, [6] which describes an anisotropic, rotating universe. We also consider other Gödel type solutions, which can give rise to a more realistic case. Basic idea is to show that if we are living in a 3-brane, then it should represent a universe with negligible rotation, even though there may be significant rotation in bulk and on the hidden brane.

**Case A:** For  $A = \rho$ ,  $B = 0$  and  $C = 0$  the solution is,

$$ds^2 = e^{-2k|y|} \left( a^2 (dt^2 - dx_1^2 + (e^{2x_1}/2) dx_2^2 - dx_3^2 + 2e^{x_1} dt dx_2) \right) + dy^2, \quad (22)$$

However this solution is not stable in this case, as proved in next section.

**Case B:** Consider a expanding, rotating and shear free universe as given in Ref. [14] on the brane with equation of state  $\hat{p} = -\hat{\rho}$ . This solution gives rise to inflationary scenario in presence of rotation. In ref. [14], it is shown that vorticity decreases exponentially as universe expands exponentially. If we embed this universe on the brane, the solution takes the form,

$$ds^2 = e^{-2k|y|} [(dt + A(t)w^1)^2 - B^2(t)((w^1)^2 + (w^2)^2 + (w^3)^2)] + dy^2 \quad (23)$$

where expansion is generated by time dependent scale factors  $A(t)$  and  $B(t)$ , and  $w^1, w^2, w^3$  are one forms

$$w^1 = -\sin x^1 dx^1 + \sin x^1 \cos x^3 dx^2, \quad (24)$$

$$w^2 = \cos x^3 dx^1 + \sin x^1 \sin x^3 dx^2, \quad (25)$$

$$w^3 = \cos x^1 dx^1 + dx^3. \quad (26)$$

Effective 4-D Einstein equation is given by,

$$R_{\mu\nu}^{(4)} = -\Lambda \hat{g}_{\mu\nu}, \quad (27)$$

where  $\Lambda$  is the 4-D cosmological constant. Solution of Einstein equation, gives [14]

$$A(t) = A_0 \sinh^{1/2}(H_0 t) \operatorname{sech}(H_0 t) \quad (28)$$

and

$$B(t) = \frac{1}{2H_0} \cosh(H_0 t), \quad (29)$$

where  $H_0 = \sqrt{\frac{\Lambda}{3}}$  and  $A_0$  is a constant. Vorticity is defined by [14]

$$V^\mu = -\frac{1}{2} \eta^{\mu\nu\alpha\beta} u_{\nu;\alpha} u_\beta, \quad (30)$$

where  $u^\mu$  is the four velocity vector of the cosmic fluid,  $u_{\nu;\alpha}$  is the covariant derivative of the four velocity vector of the cosmic fluid and  $\eta$  is the completely antisymmetric tensor. Expansion rate is given by [14]

$$H = 3 \frac{\dot{B}}{B} = 3H_0 \tanh(H_0 t), \quad (31)$$

and first component of vorticity is

$$V^1 = \frac{1}{2} \frac{A}{B^2}. \quad (32)$$

In next section we show that stabilization of extra dimension sets the time scale at which inflation takes place. Duration of inflation determines the decay in vorticity.

**Case C:** Static Gödel metric in the presence of perfect fluid source other than  $p = 0$  Gödel solution derived in [15] can be embedded on the 3-brane. Solution is given by in terms cylindrical coordinates

$$ds^2 = e^{-2k|y|} \left( dt^2 + \mu^2(\gamma - 1)^2 d\phi^2 - \sigma^2 d\phi^2 - dr^2 - dx_3^2 + 2\mu(\gamma - 1) dt d\phi \right) + dy^2, \quad (33)$$

where  $\gamma = \cosh r$ ,  $\sigma = \sinh r$ , and  $\mu$  is the rotation parameter in the theory. These solutions represent a rotating universe in presence of electromagnetically charged perfect fluids for  $\mu^2 > 2$  [15]. In the next section we will derive the constraint on  $\mu$  by stabilizing the size of the extra dimension.

## 4 Stabilization of the Extra Dimension

Stabilization of size of the extra dimension can be achieved by minimizing the effective four dimensional curvature on the brane, by integrating over extra dimension. The five dimensional curvature is given by

$$R = -\frac{1}{6M^3} \left( g^{AB}T_{AB} - 5\Lambda - 4(v_1\delta(y) + v_2\delta(y - y_c)) \right) \quad (34)$$

For the general metric (7)

$$g^{AB}T_{AB} = -A + 3B + C + P = 3P \quad (35)$$

Five dimensional action is given by

$$S_{5D} = \int d^4x \int_{-y_c}^{y_c} 2M^3 \sqrt{-g} R dy \quad (36)$$

Substituting Eq. (34) and Eq. (35) into Eq. (36) and after integrating over  $y$ -coordinate, we get

$$S_{5D} = \int d^4x \sqrt{-\tilde{g}} \frac{-1}{3} \left( -\frac{6M^3}{k} R_{(4)} (1 - e^{-2ky_c}) - \frac{5}{2k} \Lambda (1 - e^{-4ky_c}) - 4v_1 - 4v_2 e^{-4ky_c} \right) \quad (37)$$

Using the relation among  $\Lambda$ ,  $v_1$  and  $v_2$ , effective four dimensional curvature becomes

$$\tilde{R}_{(4)} = \frac{6M^3}{k} R_{(4)} (1 - e^{-2ky_c}) + 36M^3 k (1 - e^{-4ky_c}). \quad (38)$$

$\tilde{R}_{(4)}$  is the effective four dimensional curvature. One can easily see from the Eq. (38) that effective 4-D curvature has a minimum only when  $R_{(4)}$  becomes negative. By differentiating with respect to  $y_c$  we get

$$e^{-2ky_c} = -\frac{R_{(4)}}{12k^2}. \quad (39)$$

**Case A:** In this case  $R_{(4)} = 1/a^2$  is positive and  $\tilde{R}_{(4)}$  does not have a minimum.

**Case B:** In this case  $R_{(4)} = 4\lambda$ . Eq. (39) leads to

$$\Lambda = 3k^2 e^{-2ky_c}. \quad (40)$$

If we take  $\Lambda = 3k^2 \times 10^{-32}$  then we get  $ky_c \approx 40$  which solves hierarchy between plank scale and electroweak scale. If we take  $k = M_{pl}$ ,  $\Lambda \approx 10^{-60} sec^2$ , than inflation should occur at  $10^{-30}$  second in contrast to GUT time  $10^{-35}$  second. Expansion rate given in Eq. (31) corresponds to de-Sitter universe. Inflation ends at  $\sim 10^{-28}$  seconds in our case. During inflationary period vorticity decreases by an order of  $10^{-55}$  which implies that effects of rotation are diluted by inflation. This is effectively a transition from a rotating and expanding universe, to an expanding universe with little rotation.

**Case C:** In this case four dimensional curvature is

$$R_{(4)} = -\frac{(\mu^2 - 4)}{16\pi G_4}. \quad (41)$$

If  $\mu^2 = 4$ , we obtain a RS2 model [12] where  $y_c \rightarrow \infty$ , i.e. infinite fifth dimension. If  $\mu^2 > 4$ , relation between the rotation parameter and the size of the extra dimension becomes,

$$e^{-2ky_c} = \frac{(\mu^2 - 4)}{8\pi G_4} \frac{1}{24k^2}. \quad (42)$$

If  $\frac{(\mu^2 - 4)}{8\pi G_4} \frac{1}{24k^2} = 10^{-32}$ , then we get  $ky_c \approx 40$ , which is the value needed to solve the hierarchy problem. This value is also obtained by another stabilization mechanism [16].

## 5 Discussion

For  $k \sim M_{pl}$ , the term  $(\mu^2 - 4)$ , which determines the strength of the rotation in this universe (Case C), turns out to be order of  $10^{-32}$ . This is too small to show any appreciable effect of rotation on the visible brane. The vanishingly small value of rotation parameter as calculated for RS model in this paper, and other estimates of angular velocity (indicated in introduction) therefore suggest the conclusion, that if our universe lives on a 3-brane explained by RS1 model then universe must be a non-rotating one, without any anisotropy. On the other hand, if the anisotropic models are correct, than RS models must require additional input. Detailed investigation also needs to be done in case of a rotating and expanding Gödel type universe [17] embedded as a brane in RS models.

## References

- [1] Barrow, J. D. and Dabrowski M. P.: *Phys. Rev. D*, **58**, 103502 (1998). gr-qc/9803048.
- [2] Kanti, P. and Vayonakis, C.E.: *Phys. Rev. D*, **60** (1999) 103519. gr-qc/9905032.
- [3] Carrion, H.L., Reboucas, M.J. and Teixeira, A.F.F.: *J. Math. Phys.*, **40** (1999) 4011-4027. gr-qc/9904074.
- [4] D. Astefanesei, R. Mann and E. Radu, hep-th/0406050.
- [5] D. Astefanesei, R. Mann and E. Radu, *JHEP*, **0501**, 049 (2005). hep-th/0407110.
- [6] Gödel K.: *Rev. Mod. Phys.*, **447**, 447 (1949).
- [7] Birch, P: *Nature*, **298**, (1982), 451.
- [8] Jain, P.K., and Ralston, J.P.: *Mod. Phys. Lett.*, **A14**, (1999), 417.
- [9] Ralston, J.P., Jain, P.K.: *Int. J. Mod. Phys.*, **D13**, (2004), 1857-1878.
- [10] Antoniadis, I., Arkani-Hamed, Dimopoulos, N., and Dvali, G.: *Phys. Lett. B*, **436**, 257 (1998).
- [11] Randall, L. and Sundrum R': *Phys. Rev. Lett.*, **83**, 3370 (1999).
- [12] Randall, L. and Sundrum, R.: *Phys. Rev. Lett.*, **83**, 4690 (1999).
- [13] Hongya Liu and Guowen Peng: *Gen. Rel. Grav.*, **34**, 1579 (2002).
- [14] Gron, O.: *Phys. Rev. D*, **33**, 1204 (2002).
- [15] Antonio Enea Romano and Charles Goebel: *Gen.Rel.Grav.*, **35**, 1857 (2003), gr-qc/0208093.
- [16] Goldberger, W. D. and Wise, M. B.: *Phys. Rev. Lett.* **83**, 4922 (1999).
- [17] Obukhov Y. N., astro-ph/0008106.