Recurrence metrics and time varying light cones

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Abstract

It is shown by explicit construction of new metrics, that General Relativity can solve the exact Poincáre recurrence problem. In these solutions, the light cone, flips periodically between past and future, due to a periodically alternating arrow of the proper time. The geodesics in these universes show periodic Loschmidt's velocity reversion $v \rightarrow -v$, at critical points, which leads to recurrence. However, the matter tensors of some of these solutions exhibit unusual properties - such as, periodic variations in density and pressure. While this is to be expected in periodic models, the physical basis for such a variation is not clear. Present paper therefore can be regarded as an extension of Tipler's "no go theorem for recurrence in an expanding universe", to other space-time geometries.

KEY WORDS: General Relativity, Poincáre Recurrence, Black holes, White holes, Closed Time-like Curves.

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1 Introduction

The concept of recurrence metrics, introduced in this paper derives its motivation from the Poincáre recurrence problem, i.e., given a system of N, particles, under what conditions, will the system return to its initial configuration in the phase space? In an unbounded flat Newtonian space-time, non-interacting particles move on straight lines, and any initial configuration would never recur. Recurrence is also inconceivable in interacting case in a Newtonian universe, given the usual physical potentials for particle interactions. Curved space-times, with closed geodesics, however offer another approach to the Poincáre recurrence problem. On a space such as S^n, or P^n, any set of non-interacting particles, all having an identical uniform velocity v, the initial configuration would recur after a period $T = vC$, where C is the circumference of these closed (S^n or P^n) spaces. S^n and P^n are termed as example of Zoll phenomena [1], i.e. closure of all geodesics in an identical period. A Lorentzian manifold, all of whose null geodesics are closed, is said to have a Zollfrei metric [2]. Examples of Zollfrei metric are $S^n \times S^1$ and $P^n \times S^1$ space-times. Here S^1 is the time topology, and S^n and P^n are topologies of spatial sections. These models are cyclic in the sense, that each event has an infinite number of copies in past and future. A set of non-interacting photons in such universes, would circle the closed universe in a single cycle, and the initial configuration would recur. Such compact space-times offer freedom from infra-red divergences of quantum field theory [3, 4]. $S^3 \times S^1$ is also the basis of Segals [5] cosmological model. However, particles slower than light would take longer to circum-navigate the closed universe. An initial configuration of particles, with different initial velocities, e.g. a Boltzmann distribution, would therefore not recur. Hence, we are lead to the question, whether there exist general relativistic metrics, where recurrence occurs for a wider set of initial data, i.e., the complete range of particle velocities. It would also be desirable if this occurs for interacting case as well. as free particles move along geodesics, therefore, for non-interacting particles, we require equi-period closure all geodesics, for recurrence. In interacting case - assuming the interactions to be point-like, the particles will be switching from one geodesic to another. The particle paths therefore, can be described as piece wise geodesic. In this paper we present simple solutions of Einsteins field equations, where the recurrence of initial configuration occurs, due to equi-period closure of geodesics..

While spatial variation of light cone semi angle is well know, in general relativity, e.g., the Schwarzschild, and Gödel [6] metrics, but periodic, temporal variation of light cone semi angle has not received much attention. These solutions depict, a non-uniform flow of time, both temporally as well as spatially depending upon the metric, to the extent that flow of 'proper' time 's' gets reversed during half of 'coordinate' time 't'. In these solutions, time is ascribed the topology S^1/Z_2 . Such space-times of Lorentzian signature can be constructed by taking geometric product of this topology of time, with 3-spaces, which could be the spatial section of space-times such as, Minkowski, Schwarzschild. A new class of geodesic closed curves, which are almost everywhere time-like (except at critical reversal points), is obtained in the process. Such curves may be termed as Closed Time-like Curves, almost everywhere (CTCs,a.e.). We use the term recurrence metrics to describe such space times, as, any initial distribution of classical point particles, recurs after a period 2π. Matter tensors of some of these solutions are found to possess unusual properties, such as periodic variation of pressure and density. While this would be expected in periodic models, the physical basis for such a variation is not clear. The paper therefore can be regarded as an extension of Tipler's [7] "no go theorem for recurrence in an expanding universe", to other space-time geometries.

2 Recurrence metrics

Consider a space-time in which time is having S^1/Z_2 topology. This may be obtained from the Minkowski Universe as follows. First identify the hypersurfaces $t = 0$, and $t = 2\pi$. This assigns a S^1 topology to time. Next identify end points of chords of S^1, which makes topology of time S^1/Z^2. This is essentially the set of chords of the circle. Line element for this space-time - which we call the 'Periodic Minkowski' metric, is obtained from Minkowski metric by the subsitution,

$$t \rightarrow \sin t \tag{1}$$

This gives the following line element,

$$ds^2 = \cos^2 t dt^2 - dr^2 \tag{2}$$

where,$r^2 = x_1^2 + x_2^2 + x_3^2$, and (x_1, x_2, x_3) are rectangular co-ordinates.

2.1 Consistency between Global Topology, and Geodesics

Periodic topologies of time such as S^1 and S^1/Z_2, require recurrence that Cauchy data at hypersurfaces t and $t+2\pi$, be identical. If $q_n(t)$ and $p_n(t)$ represent position and momentum respectively, of n-th particle at time instant t, then recurrence requirement implies,

$$(q_n(t), p_n(t)) = (q_n(t+2\pi), p_n(t+2\pi)) \tag{3}$$

for all n. Since non-interacting particles move along geodesics, therefore, consistency between a compact topology of time and geodesics, requires that all geodesics close in same period 2π. Further, as geodesics are derived from metric, via the geodesic equation,

$$\frac{d^2x^\mu}{ds^2} + \Gamma^\mu_{\rho\sigma}\frac{dx^\rho}{ds}\frac{dx^\sigma}{ds} \tag{4}$$

the recurrence constraint has to be imposed through the metric. We show in subsection on Geodesics Equations (below), that geodesics of metric (2), have a global geometry consistent with the compact S^1/Z_2 time topology, i.e., all of them close in an identical period 2π. If on the other hand we transform metric (2) to,

$$ds^2 = dt^2 - dr^2 \tag{5}$$

and impose compact time topology by fiat - i.e., identifying the hypersurfaces $t = 0$, and $t = 2\pi$, then the geodesic equations, yield the following solution to particle motion,

$$x_i(t) = C_i^1 t + C_i^2 \tag{6}$$

for spatial coordinates x_i, $i = 1, 2, 3$. Here C_i^1 and C_i^2, can be interpreted as the initial velocity, and the initial position, respectively, of the particle. Its clear that the geodesics of (5) do not close. Thus the global geometry of (5) is inconsistent with the imposed periodic topology of time. How do we resolve this conundrum? The answer offered by author is that metrics (2) and (5) correspond to two different space-times, which we label M_1 and M_2 respectively. M_2 should be regarded as universal covering space of M_1. Topology of time in M_2, should be R^1, for it to be consistent with global geometry of its geodesics. It should be kept in mind, that where as metric is a local specification of space-time geometry, geodesics are global objects, which relate to space-time's global topology. Thus while locally M_1 and M_2 have identical geometries, however, globally geometry of their geodesics differ.

2.2 Proper time

For a fixed value of r, we have for proper time s,

$$s = \int \cos t dt \tag{7}$$

Notice that proper time s is not a monotonically increasing function, but periodically reverses direction. As coordinate time t varies monotonically between $[0, 2\pi]$, proper time s oscillates between 1 and -1. Note that a system described at time t, and undergoing time reversals at instants $t+\pi$ and $t-\pi$ will also show recurrence. But here the system undergoes discontinuous changes at moments of time reversals. On the other hand in metric 2, the changes are smooth.

2.3 Light cone

For light cone we have following relation between r and t,

$$\frac{dr}{dt} = \cos t. \tag{8}$$

The behavior of light cone semi angle is same as that of the function $\cos t$. The physical significance of such a time varying light cone semi angle however has interesting implication. dr/dt decreases in the interval $[0, \pi]$, becoming zero at $t = \pi/2$ - indicating a gradual slowing down of all particles - including light - with an instantaneous stop at $t = \pi/2$. Further, dr/dt is negative in the interval $[\pi, 3\pi/2]$, indicating a reversal of direction of propagation of all particles, and another stoppage at $t = \pi$. dr/dt becomes positive once again, in the interval $\pi/2, 2\pi$, with implies particles start moving in their initial direction, gradually gaining speed.

2.4 Loschimdt velocity reversion

The points where dr/dt becomes zero are termed as critical points in Morse theory [4], for functions defined on circles. It has been pointed out, that one of the conditions for Poincáre recurrence, is the Loschimdt velocity reversion, with velocity $v \to -v$, simultaneously for all particles. As a result of this velocity reversion, all particles would retrace their paths backwards, leading to recurrence of initial position, but not the initial velocity. For recurrence of initial velocities also, the particles are allowed to go beyond their initial positions, and a second velocity reversion is applied. This subsequently leads to complete Poincáre recurrence of both positions as well as velocities. Such phenomena ordinarily do not occur in physical situations, - but may be achieved by applying time reversal to the system at instants π and π. The smoothly time varying light cones suggested here, allow this to happen smoothly. We verify in sub-sections on Einstein's equations and Matter tensor that metric in eq. (2) is a solution of Einstein's field equations. Below we verify periodicity of geodesic equations.

2.5 Geodesic Equations

The only non-zero Christoffel symbol is -

$$\Gamma^0_{00} = -\tan t \tag{9}$$

Substituting this in the geodesic equation gives following equations for t and x_i

$$\begin{aligned} t''(s) - t'(s)\tan t(s) = 0, \\ x_i''(s) = 0, i = 1, 2, 3, \end{aligned} \tag{10}$$

with solution of the form

$$\begin{aligned} t(s) = \arcsin[C^1_0(s - C^2_0)] \\ x_i(s) = C^1_i s + C^2_i, \quad i = 1, 2, 3. \end{aligned} \tag{11}$$

where, $C^1_\mu, and C^2_\mu, \mu = 0, 1, 2, 3,$ are integration constant. This yields the following relation for spatial coordinate $x_i(t)$, and velocity $v_i(t) = x_i'(t)$, as periodic function of time t.

$$\begin{aligned} x_i(t) = \frac{C^1_i}{C^1_0}(\sin t + C^2_0) + C^2_i \\ v_i(t) = \frac{C^1_i}{C^1_0}\cos t \end{aligned} \tag{12}$$

Integration constants $C^1_\mu, and C^2_\mu$, can be interpreted in terms of initial position $x0$ and initial velocity $v(0)$ as follows -

$$x_i(0) = \frac{C^1_i C^2_0}{C^1_0} + C^2_i$$

$$v_i(0) = \frac{C_i^1}{C_0^1} \tag{13}$$

which gives 8 unknown integration constants in terms of 6 initial conditions. Setting

$$\begin{aligned} C_0^1 &= 1 \\ C_0^2 &= 0 \end{aligned} \tag{14}$$

gives $s = \sin t$ and allows determination of remaining integration constants in terms of initial conditions.

2.6 Einstein equations

It can be verified that Riemann tensor vanishes,

$$R_{\mu\nu\rho\sigma} = 0, \tag{15}$$

and therefore Ricci tensor $R_{\mu\nu}$, Ricci scalar R, and Einstein tensor $G_{\mu\nu}$ also vanish. Einstein equation with cosmological constant $\Lambda = 0$ is,

$$G_{\mu\nu} = 0 = T_{\mu\nu}, \tag{16}$$

and for non-zero Λ it is

$$\Lambda \cos^2 t = T_{\mu\nu} \tag{17}$$

where, $T_{\mu\nu}$ is the matter tensor. Following interpretations for matter tensor are possible.

2.6.1 $\Lambda = 0$

1. A vacuum solution with zero matter density ρ and zero pressure p. This however, is not interesting from recurrence view point, as recurrence is supposedly for a system populated with particles.

2. A solution with classical particles of zero mass $m = 0$, and therefore carrying zero momentum, and exerting zero pressure. Note that quantum particles with $m = 0$ such as photons will contribute to universe's matter density, and therefore do not form a part of this solution.

3. A solution with equal number of classical particles with positive mass m, and negative mass $-m$. This would ensure zero matter density at a macroscopic (coarse grained) scale. We note that negative energy states in general relativity also occur in other situations such as worm holes. Properties of negative mass particles are discussed in [10]. Equal amounts of Tachyonic matter of form im and $-im$ would also be admissible. Interaction between all these particles should be such as to ensure zero pressure on macroscopic scale. Note that Dirac's interpretation for the positive and negative energy solutions of Klein-Gordon equation allows equal number of positive matter and negative energy anti-matter. Whether universe is matter-antimatter asymmetric or not has been a source of debate. It has been suggested that one of possible origins for high energy cosmic rays is matter-antimatter annihilation occurring in the universe.

2.6.2 $\Lambda \neq 0$

This yields,

$$\begin{aligned} \rho &= \Lambda(\cos^2 t + 1) \\ p &= -\Lambda \end{aligned} \tag{18}$$

So either the pressure is negative or density is negative - depending upon the sign of Λ.

From above possibilities for matter tensor, we see that there are severe constraints on matter states and inter-particle interaction [1] for recurrence in general relativity. Behavior of density in particular in eqs. (18) needs physical explanation. A qualitative attempt for this is made in section on "Speculation on periodic mass acquisition" in particle models.

2.7 Wave equation

The wave equation for $1-D$ form of (2) is

$$\frac{1}{\cos^2 t}\frac{\partial^2\phi}{\partial t^2} - \frac{\partial^2\phi}{\partial x^2} = 0 \tag{19}$$

with the solution

$$\phi = \exp^{[kx-\omega\sin t]} + \exp^{-[kx-\omega\sin t]} \tag{20}$$

[1]Sign of pressure depends upon whether inter-particle forces are attractive or repulsive.

where the first term represents the retarded component and the second term represents the advanced component, of the solution. It can be seen, that advanced wave differs from retarded wave only by a phase π. Usually the advanced waves are considered as time reversed retarded waves. However, in the (20), advanced waves can be interpreted as retarded waves, returning to a point after a time interval of π.

2.8 Causality violation

There exist a number of solutions of general relativity having Closed Time-like Curves (CTCs). Gödel [6] was the first one in which existence of CTCs was demonstrated. The metric presented here, generates a closed curve in space-time, which consists of two time-like segments joined at ends. However, this curve is space-like at points of join. The closed curve thus generated can hence be termed as a "CTC, almost everywhere" (CTC, a.e.).

2.9 Background fluctuation

Random background fluctuations will destroy recurrence. However, fluctuations with time periodic constraint, (i.e., which can be expanded as a commensurate Fourier series) will preserve recurrence.

3 Periodic Schwarzschild type universes

The maximal extension of Schwarzschild line element, in a form without coordinate singularity [11] is,

$$ds^2 = f^2 dt^2 - f^2 dr^2 - r^2 d\theta^2 - r^2 \sin^2 \theta d\phi^2 \tag{21}$$

where,

$$f^2 = \frac{32Gm^3}{r} \exp\left(-\frac{r}{r_s}\right) \tag{22}$$

Consider the modified Schwarzschild line element in which t is compactified to S^1/Z_2 topology, as in the periodic Minkowski case. The line element is now obtained from eq.(21) by the using the substitution of eq.(1), which gives,

$$ds^2 = f^2 \cos^2 t dt^2 - f^2 r^2 - r^2 d\theta^2 - r^2 \sin^2 \theta d\phi^2 \tag{23}$$

Here R_s is the Schwarzschild radius. It can be verified that eq.(23)is a Ricci flat metric and therefore constitutes a vacuum solution of Einstein's field equations. Light cone semi angle is same as in the Periodic Minkowski case. Space-time is now bounded in past ($t = -\pi/2$) and future ($t = \pi/2$), which are points of reversion. $g_{00} = 0$ while g_{11} remains finite at these points of reversion. Space-time reduces to 3 spatial dimensions at these points. g_{00} and g_{11} are finite at $r = r_s$. At $r = 0$ and $t = \pm\pi/2$, $g_{00} = 32Gm^3$, is finite, while g_{11} is infinite.

This metric, however does not allows radiation and massive particles to escape to infinity because of periodic time reversals. Similarly particles with initial position r_0 and radially inward velocity $v(t)$ satisfying

$$r_0 > 2\int_0^1 v(t)dt \tag{24}$$

will not reach the Schwarzschild radius and continue to oscillate outside the horizon, due to time reversals. The limits of above integral correspond to the interval in which t is positive, i.e., $t \in [0, 1]$. During the time reversed phase, the black hole will behave as a white hole, as the particles would be coming out of the horizon. They however will not escape to infinity (again because of time reversal), and will oscillate around the horizon.

It is expected that our universe is governed by a uniform arrow of time. Can space-time (23), with a locally oscillating time arrow, be embedded in a universe, with asymptotically, uniform, time arrow? The answer is provided by the following substitution for time co-ordinate t, outside horizon,

$$t \to t + e^{[r_s - r](\sin t - t)} \tag{25}$$

This choice of t gives a behavior as $\sin t$, near the horizon, and the usual t behaviour for $r >> r_s$. A further transformation

$$t \to t^\gamma \tag{26}$$

allows periodicity to increase or decrease with time, depending upon sign of γ.

4 Discussion

Periodic Minkowski is singularity free and has a admissible vacuum solution. It also admits a solution with a universe having an equal number of positive and negative masses, which however is un-physical. The solution of Periodic Minkowski with non-zero density has un-physical periodic variation in density. Periodic Schwarzschild type metric admits a vacuum solution. Present work can be regarded as a continuation of exploration by Tipler [7].

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