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Rotating Brane World Black Holes

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ABSTRACT

A five dimensional rotating black string in a Randall-Sundrum brane world is considered. The black string intercepts the three brane in a four dimensional rotating black hole. The geodesic equations and the asymptotics in this background are discussed.

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Unification of gravity with the other fundamental interactions in the context of string theory have suggested the possibility of our universe being higher dimensional. The extra spatial dimensions in these scenarios are assumed to be compactified and hence unobservable at low energies. Recent advances have however shown that if the matter fields are localized on a three-brane (or a smooth domain wall) and the electroweak scale assumed to be the fundamental scale then the extra dimensions are not restricted to be small [1]. This removes the hierarchy between the electroweak scale and the Planck scale but introduces a new hierarchy of scales amongst the large and the small dimensions. This construction has opened the way for interesting phenomenological possibilties of low scale quantum gravity effects which may be accessible in the next generation accelarators. Recently Randall and Sundrum (RS) [2] considered a five dimensional modified version of these models in which the matter fields were restricted on a three brane with the brane coordinates being dependent on the location in the fifth dimension. These compactification geometries have been termed *warped compactification*. The models also require a regulator three brane at a certain distance in the fifth dimension. For this construction to be an acceptable solution of the five dimensional Einstein equations the three brane(s) is required to be embedded in a slice of five dimensional Anti de-Sitter(AdS) space-time. Interestingly in this model four dimensional gravity arises due to the localized zero-mode of the five dimensional Kaluza-Klein graviton. This leads to an interesting resolution of the hierarchy problem as the extra fifth dimension is not required to be large in this case. It could be shown later that the regulator brane in this scenario could be removed to infinity leading to a model with effectively a single three brane and an infintely large extra fifth dimension. The insights gained from these constructions have resulted in intense activity in this area in the recent past. A full non linear treatment in the framework of supergravity [3] have confirmed the conclusions resulting from the linearized approximation. In particular, these constructions have led to the possibility of detecting low scale quantum gravity effects on phenomenlogy at the weak scale in the next generation particle acclerators [4]. The cosmological consequences of these and related models are also being pursued.

A related direction of investigation has been the study of four dimensional black holes on the three-brane from this *brane world* perspective. In an interesting article Chamblin, Hawking and Reall(CHR) [5] have constructed a Schwarzschild black hole on the four dimensional world volume of the three brane from the five dimensional RS scenario. Such a configuration is extended in the extra dimensions and will be a higher dimensional object in the brane world. such a solution would presumably describe the end state of gravitational collapse of non-rotating matter on the three brane. The metric on the brane in this case is required to be a Schwarzschild metric to satisfy the usual astrophysical observations and the requirements of the singularity theorems. However, the obvious choice of an AdS-Schwarzschild solution in five dimensions fails to satisfy the Israel junction conditions at the location of the brane which are required to be Z_2 reflection symmetric. CHR considered an alternative configuration of a five dimensional black string which intercepted the three brane in a four dimensional Schwarzschild metric. This has the usual Schwarzschild singularity on the brane and is extended along the transverse direction. However it is observed that the solution is singular at the AdS horizon far away from the brane. A study of the geodesics show the that this singularity is actualy a parallely propagated (p-p) curvature singularity [3,8]. The solution is unstable far away from the brane due to the Gregory-Laflamme instability [9] and is conjectured to evolve to a black cigar solution which looks like a black string far away from the AdS horizon but closes off before reaching it. The exact metric for this cigar solution is yet to be determined. It was however pointed out in [10] through an explicit calculation that in these asymptoticaly AdS solutions there would be an accumulation of mini black holes towards the AdS horizon which do not indicate either a cigar or a pancake geometry in the bulk. Attempts to consider the off-brane metric in the linearized framework have also appeared in [11]. Investigation of similar constructions in a $3 + 1$ dimensional RS model with two branes, where exact solutions in the form of AdS C metrics are available, show that the corresponding black hole solutions are non-singular at the AdS horizon [12]. Generalizations of this construction for higher dimensional brane worlds [13] and charged black holes in the RS models have been obtained [14]. Numerical studies for the off brane cigar metrics have also been performed [15].

In this brief report we construct a four dimensional Kerr black hole on the three-brane in a five dimensional RS brane world scenario. It is apparent that a AdS-Kerr solution in five dimensions will fail to satisfy the junction conditions at the three-brane. The reason being that the AdS-Kerr solution reduces to the AdS-Schwarzschild solution for the rotation parameter going to zero, which fails to satisfy the junction conditions appropriate to a vaccuum solution [5]. Thus, we consider a rotating black string in five dimensions which can be shown to intercept the three brane in a four dimensional Kerr metric. We should mention here that the usual rotating black holes in the brane world and the BTZ variants have also been obtained in [12] for two branes embedded in a (3+1) dimensional AdS space-time where exact AdS C metrics are known. However such a construction is unavailable in five or higher dimensions where exact solutions are not apparent and it is neceesary to work in the framework of linearized gravity. This construction also serves as a prelude to obtaining the four dimensional Kerr-Newman solutions in a brane world scenario. Such higher dimensional constructions will also be relevant to general string compactifications. We find that like the Schwarzschild case the Kerr solutions are also singular at the AdS horizon as expected in the linearized approximation. Study of the geodesic equations clearly illustrate that the singularity at the AdS horizon is accessible only on bound state orbits. This indicates the existence of parallely propagated (pp) curvature singularities at the AdS horizon in this metric also.

The Randall-Sundrum models are based on a five dimensional AdS as the bulk space-time with the metric

$$ds^2 = \frac{l^2}{z^2}(\eta_{ij}dx^i dx^j + dz^2) \quad (0.1)$$

where $z = 0$ and $z = \infty$ are the conformal infinity and the AdS horizon respectively. Here (i, j) runs over the four dimensional world volume of the three brane and l is an AdS length scale. The actual RS geometry is obtained by slicing off the small z region at $z = z_0$ and glueing a copy of the large z geometry. The resulting topology is essentialy $R^4 \times \frac{S^1}{Z_2}$. The discontinuity of the extrinsic curvature at the $z = z_0$ surface correponds to a thin distributional source of stress- energy. From the Israel junctions conditions this may be interpreted as a relativistic three brane with a corresponding tension [7]. Another variant of this model is to slice the AdS space-time both at $z = 0$ and $z = l$ and insert three branes with Z_2 reflection symmetry at both the surfaces. The Israel conditions now require a negative tension for the brane at $z = l$. The first version may be obtained from the second by allowing the negative tension brane to approach the AdS horizon at $z = \infty$ however a dynamical realization of this is not yet clear. We will focus our considerations on the first variant of the RS geometry in this case but state that our construction should be trivialy generalized to the second variant also. The interesting consequence of the RS construction is that perturbations of the five dimensional metric are normalizable modes peaked at the location of the brane. In particular the zero mode satisfying the vaccuum Einstein equation yields standard four dimensional gravity on the brane.

The Einstein equations in five dimensions with a negative cosmological constant continue to be satisfied for any metric g_{ij} which is Ricci flat. CHR [5] considered the gravitational collpase of non-rotating matter trapped on the brane to form a black hole. The metric on the brane in this case must be a Schwarzschild metric which is Ricci flat. However the obvious choice of a five-dimensional AdS-Schwarzschild metric fails to satisfy the Israel junction conditions compatible with the Z_2 reflection symmetry. CHR [5] took the natural choice of the four dimensional metric in eqn (1) to be Schwarzschild, thus arriving at the metric which describes a five dimensional uncharged black string, in the RS geometry. The black string metric is given by,

$$ds^2 = \frac{l^2}{z^2}[-U(r)dt^2 + U^{-1}(r)dr^2 + r^2(d\theta^2 + sin^2\theta d\phi^2) + dz^2], \quad (0.2)$$

where $U(r) = 1 - \frac{2M}{r}$. Inclusion of a three brane with reflection symmetry now requires the brane tension to be compatible with the junction conditions relating the extrinsic curvatures on both sides. This gives

$$T_3 = \pm\frac{6}{\kappa^2 l}. \quad (0.3)$$

where $\kappa^2 = 8\pi G_5$ where G_5 is the five dimensional Newton's constant. The single brane model however will correspond to a positive brane tension. For the brane at

$z = z_0$ one introduces the coordinate $w = z - z_0$ and the metric is then given as

$$ds^2 = \frac{l^2}{(\mid w \mid + z_0)^2}[-U(r)dt^2 + U(r)^{-1}dr^2 \tag{0.4}$$

$$+r^2(d\theta^2 + sin^2\theta d\phi^2) + dw^2] \tag{0.5}$$

with $-\infty < w < \infty$ and the brane at $w = 0$. The metric on the brane may be recast into the standard Schwarzschild form by rescaling the t and r coordinates. The ADM mass of the black hole to an observer on the brane is then $\tilde{M} = M\frac{l}{z_0}$ and the proper horizon radius is $2\tilde{M}$. The square of the curvature scalar is given as

$$R_{\mu\nu\rho\sigma}R^{\mu\nu\rho\sigma} = \frac{1}{l^4}\left(40 - \frac{48M^2z^4}{r^6}\right), \tag{0.6}$$

where μ, ν goes over the five dimensions. This diverges at the AdS horizon $z = \infty$ and posesses the usual black string singularity at $r = 0$. However an examination of the geodesic equations in this space-time show that curvature invariants diverge only for the bound state geodesics that encounters the $r = 0$ singularity but are finite along non-bound state geodesics which reach $r = \infty$. In [5] CHR examines the curvature components in an orthonormal frame parallely propagated along a non-bound timelike geodesic. It is observed that some of the curvature components in this frame diverges at $z = \infty$. This signals the presence of parallely propagated curvature singularities [3, 8, 16] arising out of tidal effects at the AdS horizon. CHR further argue that the black string metric is unstable due to the standard Gregory-Laflamme instabilty near the AdS horizon where the black string behaves as if it is in an asymptoticaly flat space-time provided the horizon radius is small. But the AdS space acts like a confining box preventing development of perturbations with wavelengths larger than the AdS length scale so instabilities must occur at shorter wavelengths hence at large values of z. This insatbility may cause the black string horizon to pinch off before the AdS horizon leading to a bulk metric describing a black cigar. The metric on the brane being a stable Schwarzschild metric far away from the AdS horizon. Careful consideration of the black string instability in AdS spaces [10] however seems to indicate an accumulation of mini black holes towards the AdS horizon.

We generalize the construction of CHR [5] to study the occurrence of a rotating uncharged black hole on the three brane. Four dimensional General Relativity on the brane requires this to be the Kerr metric which is also Ricci flat. The extension to a five dimensional bulk AdS-Kerr metric fails to satisfy the junction conditions at the location of the brane consistent with a vaccuum configuration. This is because in the absence of rotation the AdS-Kerr solution will reduce to the AdS-Schwarzschild metric which has been shown to be incompatible with the junction conditions in [5]. This leads to our choice of the bulk metric as a rotating black string in five dimensions which is consistent with the junction conditions. The metric for the brane world black string in the Boyer-Lindquist coordinates is given as,

$$\begin{aligned} ds^2 &= \frac{l^2}{(z_0+|w|)^2}\left[-(\frac{\Delta - a^2sin^2\theta}{\Sigma})dt^2 + \frac{\Sigma}{\Delta}dr^2\right. \\ &+ \Sigma d\theta^2 + \frac{4aMr\sin^2\theta}{\Sigma}d\phi dt \\ &+ \left.[\frac{(r^2+a^2)^2 - \Delta a^2 sin^2\theta}{\Sigma}]sin^2\theta d\phi^2 + dw^2\right]. \end{aligned} \tag{0.7}$$

Where we have

$$\Sigma = r^2 + a^2cos^2\theta \tag{0.8}$$

and

$$\Delta = r^2 + a^2 - 2Mr. \tag{0.9}$$

Here the location of the brane is at $z = z_0$ with the Z_2 reflection symmetry imposed there. Notice that the four dimensional part of the metric is Ricci flat apart from the conformal factor and as a consequence this guarantees the satisfaction of the five dimensional Einstein equations with a negative cosmological constant. As in the case of CHR [5] the induced metric on the brane may be recast in the form of the standard four dimensional Kerr-metric by suitable rescaling. The ADM mass and the angular momentum for the rotating black hole on the brane are then given as $M^* = \frac{Ml}{z_0}$ and $J^* = \frac{l^2}{z_0^2}Ma$ to an observer confined to the brane. We assume here that $a^2 \leq M^2$ to avoid the occurrence of naked singularities [16]. The Kerr metric on the brane will exhibit the usual features of the inner and outer horizons and an ergosphere. The horizons will now be given for $r\pm = M^* \pm (M^{*2} - a^{*2})^{\frac{1}{2}}$ and the stationary limit surface at $r_s = M^* + (M^{*2} - a^{*2}cos^2\theta)^{\frac{1}{2}}$.

The square of the curvature tensor is determined as [12],

$$R_{\mu\nu\rho\sigma}R^{\mu\nu\rho\sigma} = \frac{1}{l^4}[40 + \frac{48M^2z^4}{\Sigma^6}(r^2 - a^2cos^2\theta)(r^4 - 14a^2r^2cos\theta + a^4cos^4\theta)] \tag{0.10}$$

This shows the usual ring singularity of the Kerr metric for $\rho = 0$ and $\theta = \frac{\pi}{2}$ and diverges at the AdS horizon at $z = \infty$. The ring singularity for the rotating black string has a translational symmetry in the transverse direction and is thus an example of an extended singular solution in five dimensions which is asymptoticaly AdS.

Our metric for the rotating black string being a stationary axisymmetric metric posesses two timelike Killing isometries. If u is the tangent vector to a timelike or null geodesic with an affine parameter λ, the timelike Killing vectors $\xi = \frac{\partial}{\partial t}$ and $\chi = \frac{\partial}{\partial t}$ gives rise to two conserved quantities $E = -\xi.u$ and $L = \chi.u$. Rearrangement of these equations [16] provides us with the geodesic equations for the t and ϕ directions for motion in the equatorial plane $\theta = \frac{\pi}{2}$. These turn out to be as follows

$$\frac{dt}{d\lambda} = \frac{z^2}{l^2\Delta}\left[\left(r^2 + a^2 + \frac{2a^2M}{r}\right)E - \frac{2aM}{r}L\right] \tag{0.11}$$

$$\frac{d\phi}{d\lambda} = \frac{z^2}{l^2\Delta}\left[\left(1-\frac{2M}{r}\right)L + \frac{2aM}{r}E\right] \tag{0.12}$$

The z equation is then given as

$$\frac{d}{d\lambda}\left(\frac{1}{z^2}\frac{dz}{d\lambda}\right) = -\frac{\sigma}{zl^2}. \tag{0.13}$$

Here $\sigma = 0$ for null and $\sigma = 1$ for the timelike geodesics. The solutions for null geodesics are $z =$constant or

$$z = \frac{-z_1 l}{\lambda}, \tag{0.14}$$

The solution for a timelike geodesic is

$$z = -z_1 cosec(\lambda/l). \tag{0.15}$$

The solution $z = const.$ relates to the Schwarzschild case hence we focus our attention on the other solutions which appear to reach the AdS horizon at $z = \infty$. For this class of solutions the radial geodesic equation in the equatorial plane is computed to be

$$\left(\frac{dr}{d\lambda}\right)^2 + \frac{z^4}{l^4}\left[\left(\frac{L^2-a^2E^2}{r^2} - \frac{2M}{r^3}(aE-L)^2 - E^2\right)\right] + \frac{l^2}{z_1^2}\frac{\Delta}{r^2} = 0 \tag{0.16}$$

Following CHR we now introduce a new parameter $\nu = -\frac{z_1^2}{\lambda}$ for the null geodesics and $\nu = -(z_1^2/l)cot(\frac{\lambda}{l})$ for the timelike geodesics. We also define new coordinates $\tilde{r} = z_1 r/l, \tilde{t} = z_1 t/l$, and new constants $\tilde{E} = z_1 E/l, \tilde{L} = z_1^2 L/l^2, \tilde{M} = z_1 M/l$ and $\tilde{a} = z_1 a/l$. Using these rescaled qunatities in the radial equation it is possible to remove the explicit z dependence and the equation may be recast in an equivalent four dimensional form. This matches with the radial equation for a timelike geodesic in a four dimensional Kerr black hole with a mass $\tilde{M}$ [16].

$$\left(\frac{d\tilde{r}}{d\nu}\right)^2 + \left[\frac{\tilde{L}^2-\tilde{a}^2\tilde{E}^2}{\tilde{r}^2} - \frac{2\tilde{M}}{\tilde{r}^3}(\tilde{a}\tilde{E}-L)^2 - \tilde{E}^2 + \frac{\tilde{\Delta}}{\tilde{r}^2}\right] = 0 \tag{0.17}$$

In this case ν is the proper time along the geodesic. Notice that both the timelike and null geodesic in five dimensions gives rise to timelike four dimensional geodesics.

It is now possible to study the behaviour near the singularity that is $\lambda \to 0^-$ which is equivalent to the four dimensional affine parameter $\nu \to \infty$. This describes the late time behaviour of the corresponding four dimensional geodesics. The treatment is simmilar to CHR [5]. The geodesics which reach the singularity at $\tilde{r} = 0$ do so in finite affine parameter ν. For infinite ν there are two cases. For one the geodesic reaches $\tilde{r} = \infty$ and for the other we have bound states or orbits restricted to finite $\tilde{r}$ outside the horizon. It is evident from the expression eqn.(8) for the singularity that for the orbits that reach $r = \infty$ or non bound state orbits, the curvature squared remains finite at $z \to \infty$, the AdS horizon. However, the bound state orbits at finite r encounter a curvature singularity at the AdS horizon. This is completely similar to the Schwarzschild case as in CHR where the singularity at the AdS horizon is a parallely propagated (p-p) curvature singularity. It is conceivable that in the rotating case too the singularity at the AdS horizon is also a pp singularity.

To determine this it is necessary to calculate the projection of the curvature tensor to an orthonormal frame which is parallely propagated on a non bound time-like geodesic. In the case of the non-rotating Schwarzschild metric this reduces to the problem of determining a unit normal to the tangent to a time like non bound geodesic. For this diagonal metric this becomes an effectively two dimensional problem for a specific choice of the normal. However the non-diagonal Kerr metric such a determination of the corresponding normal reduces to an effectively three dimensional exercise. This requires the solution of three coupled partial differential equations in order to satisfy the parallel transport equations and becomes computationaly intractable. However we emphasize that such an orthonormal frame should exist in which it would be possible to show that the singularity at the AdS horizon is indeed a pp curvature singularity.

Following CHR [5] it is conceivable that the rotating black string will also be subject to the Gregory-Laflamme instability [9] arising out of long wavelength perturbations. This would make the black string unstable near the AdS horizon but stable far away causing the horizon to pinch off and form a line of mini black holes. However from the considerations in [10] for Schwarschild black strings in AdS space this instability would tend to form an accumulation of mini black holes towards the AdS horizon. As shown in [10] this would result in a "sttotie" shape for the event horizon of a Schwarschild black string with masses comparable to the AdS curvature. For masses much larger than the AdS curvature the resulting solution will look like a five dimensional black hole. The arguments in [10] should generalize also for the case of the rotating black string presented here, however careful considerations are needed for the two horizons and the stationary limit surface. Presumably the instabilty would also cause the black hole to be unstable near the AdS horizon signaling the presence of the pp singularity. As shown in [12] in the 3 + 1 dimensional RS model the pp singularity at the AdS horizon seems to be an artifact of the linearized approximation. This is also borne out in the full non linear supergravity considerations in [3]

In conclusion we have obtained a description of a four dimensional uncharged rotating black hole on the brane described by a Kerr metric from a five dimensional RS brane world perspective. The bulk five dimensional solution which intercepts the three brane in a rotating black hole has been considered to be a five dimensional rotating black string in the RS geometry. This choice is compatible with the requirement of the Israel junction conditions arising out of the reflection symmetry at the location of the three brane in the RS brane world. The Kerr metric on the brane is shown to exhibit the usual ring singularity in addition to a singularity at the AdS horizon.

Furthermore we have obtained the geodesic equations in the equatorial plane for this background and observe that the singularity at the AdS horizon is accessible only on bound time like geodesics which remain at finite orbit radius. This signals the occurrence of parallely propagated (pp) curvature singularity at the AdS horizon. The construction of orthonormal frame to explicitly study the pp singularity seems non-trivial due to the non diagonal nature of the extended Kerr metric for the black string. We further state that the solution presented will be subject to the usual black string instability causing the formation of mini black holes which would accumulate towards the AdS horizon. However an explicit calculation in the spirit of [10] is necessary for the rotating black string case considered here. It is also imperative to construct an exact description of the off-brane bulk metric. This would presumably be a generaliztion of the AdS C-metric in four dimensions. It would be relevant to generalize our construction for the full Kerr-Newman metric on the brane which would describe a charged rotating black hole in a brane world. Work is in progress along these directions.

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References

[1] N. Arkani-Hamed, S. Dimopoulos , G. Dvali, Phys. Lett.B 429, 263 (1998), hep-ph/9803315; I. Antoniadis, N. Arkani-Hamed, S. Dimopoulos, G. Dvali, Phys. Lett. B436,257 (1998), hep-ph/9804398 .

[2] L. Randall, R. Sundrum, Phys. Rev. Lett. 83, 4690 (1999), hep-th/9906064; Phys. Rev. Lett.83, 3370 (1999), hep-ph/9905221 .

[3] A. Chamblin, G. W. Gibbons, Phys. Rev. Lett. 84 (2000), hep-th/9909130.

[4] J. Lykken, L. Randall, JHEP 0006,014 (2000), hep-th/9908076.

[5] A. Chamblin, S. Hawking, H. Reall, Phys. Rev. D 61, 065007 (2000), hep-th/ 9909025

[6] A. Chamblin, C. Csaki, J. Erlich, T. J. Hollowood,Phys. Rev. D 62, 044012 (2000), hep-th/000207

[7] S. Giddings, E. Katz, L. Randall, JHEP 0003, 023 (2000), hep-th/0002091; S. B. Giddings, E. Katz , MIT-CTP-3024 (2000), hep-th/0009176

[8] J. Podolsky, Class. Quant. Grav. 15, 719 (1998) , gr-qc/9801052.

[9] R. Gregory, R. Laflamme, Phys. Rev. Lett. 70, 2837, (1993) ,hep-th/ 9301052; Nucl. Phys. B 428, 399 (1994), hep-th/ 9407071; Phys. Rev D 51, 305 (1995), hep-th/ 9410050.

[10] R. Gregory, Class. Quant. Grav. 17, L125 (2000).

[11] I. Giannakis and H. C. Ren, Phys.Rev. D63 (2001) 024001, hep-th/0007053, *ibid* Phys. Rev. D63 (2001) 125017, hep-th/0010183 Phys.Rev. D64 (2001) 065015, hep-th/0103265.

[12] R. Emparan, G. T. Horowitz, R. C. Myers, JHEP 0001, 021 (2000), hep-th/9912135; R. Emparan, G. T. Horowitz, R. C. Myers, JHEP 0001, 007 (2000), hep-th/9911043 .

[13] Won Tae Kim, John J. Oh, Marie K. Oh, Myung Seok Yoon, Sogang-HEP 272-00,(2000), hep-th/0006134.

[14] N. Dadhich, R. Maartens, P. Papadopoulos, V. Rezania, Phys.Lett.B 487, 1 (2000), hep-th/0003061; I. Oda, EDO-EP-32, (2000), hep-th/0008055; A. Chamblin, H. S. Reall, H. Shinkai, T. Shiromizu, Phys. Rev. D 63 (2001) 064015, hep-th/0006134; H. Lu, C.N. Pope, Nucl.Phys. B 598 (2001) 492-508

[15] T. Shiromizu, M. Shibata, Rev. D 62 (2000) 127502,

[16] R. M. Wald , *General Relativity*, Univ. of Chicago Press Chicago, (1984), S. W. Hawking and G. F. Ellis, *Large Scale Structure of Space-Time*, Cambridge,Univ. Press, Cambridge (1973), S. Chandrasekhar, *Mathematical Theory of Black Holes*, Oxford Univ. Press (1985).