Use of Gödel Universe to Construct A New Zollfrei Metric with $R^2 \times S^1$ Topology

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Abstract

A new example of $(2 + 1)$-dimensional Zollfrei metric, with the topology $R^2 \times S^1$, is presented. This metric is readily obtained from the celebrated $(3 + 1)$- dimensional rotating Gödel universe $G_{3,1}$. This is because $G_{3,1}$ has the interesting property that, the light rays which are confined to move on the plane perpendicular to the rotation axis, return to their origin after a time period $T = \frac{2\pi}{\omega}[\sqrt{2} - 1]$ -where ω is the angular velocity of the universe. Hence by - the topological identification of pairs of points on the time coordinate, seperated by the time interval T. and droping the flat x_3 coordinate - which is directed along the rotation axis; one obtains the $(2 + 1)$-dimensional Zollfrei metric with the $R^2 \times S^1$ topology.

KEY WORDS: Gödel Universe, Zollfrei Metric, Closed Null Geodesics (CNCs), Closed Timelike Curves (CTCs), Periodic Time

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A Lorentzian manifold is said to possess Zollfrei metric, if all its null geodesics are closed [2]. Part of attraction for studying Zollfrei metrics in physical context comes from the Segal's cosmological model [4] which is based upon the $S^3 \times S^1$ topology of spacetime. Examples of manifolds with Zollfrei metric are - $S^n \times S^1$ and $P^n \times S^1$ - with commensurate radii of spatial and temporal factors. To our knowledge, all previously known examples of Zollfrei metric have compact topology for the spatial factor - i.e., S^n or P^n. Here, we construct a new example of Zollfrei metric in $(2+1)$-dimensions, with the $R^2 \times S^1$ topology. This construction is based upon the Gödel universe [1]. We shall refer to this as the Gödel-Zollfrei metric, and denoted it by $G_{2,1}^{S^1}$.

The $(3+1)$-D axisymmetric, homogenous, Gödel universe $G_{3,1}$ [1] with the topology $R^3 \times R^1$, has the line element -

$$ds^2 = a^2(dx_0^2 - dx_1^2 - dx_3^2 + \frac{e^{2x_1}}{2}dx_2^2 + 2e^{x_1}dx_0dx_2) \tag{1}$$

which satisfies the Einstein equations for a uniform matter density,

$$\rho = (8\pi G a^2)^{-1}, \tag{2}$$

and a cosmological constant,

$$\Lambda = -4\pi G\rho, \tag{3}$$

where, G is Newton's gravitational constant. The universe rotates about the x_3 axis, with the angular velocity,

$$\omega = 2(\pi G\rho)^{1/2} = (\sqrt{2}a)^{-1} \tag{4}$$

Lets denote the curved space defined by the three coordinates (x_0, x_1, x_2) as $G_{2,1}$. Now $G_{3,1}$ can be regarded as the product of $G_{2,1}$ and the flat x_3 coordinate (which has the topology R^1), i.e.,

$$G_{3,1} = G_{2,1} \times R^1 \tag{5}$$

Pfarr [3] worked out geodesic and non-geodesic trajectories for $G_{2,1}$. For particles confined to move on the (x_1x_2) plane - i.e., the plane perpendicular to the rotation axis x_3, he showed that the geodesically moving

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particles, move on circles and consequently return to the point of their origin. In particular, the light rays move on the largest circles (null geodesics), and reach out to a maximal distance r_H, from the point of their origin,

$$r_H = \frac{2}{\omega} \ln(\sqrt{2} + 1) \tag{6}$$

The circle defined by r_H, may be termed as the Gödel horizon. The circles of radius $r = r_H$ are Closed Null Curves (CNCs), while circles of radius $r > r_H$ are Closed Timelike Curves (CTCs). The time taken by the light rays to return to the point of their origin is [3] -

$$T = \frac{2\pi}{\omega}[\sqrt{2} - 1] \tag{7}$$

Accordingly if in $G_{3,1}$-

1. One drops the flat x_3 coordinate, which is directed along the rotation axis, and

2. Compactifies the time coordinate x_0, to S^1, and

3. Chooses the time period of S^1 time factor equal to $T = \frac{2\pi}{\omega}[\sqrt{2} - 1]$;

one obtains this new example of $(2 + 1)$-dimensional Zollfrei metric with the $R^2 \times S^1$ topology.

Gödel [1] gave a list of nine interesting properties of $G_{3,1}$. Remarkably, in the fifth property, he considered the possibility of both an open R^1 and a closed S^1 time coordinate.

References

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